Multivariate Geometric Anisotropic Cox Processes

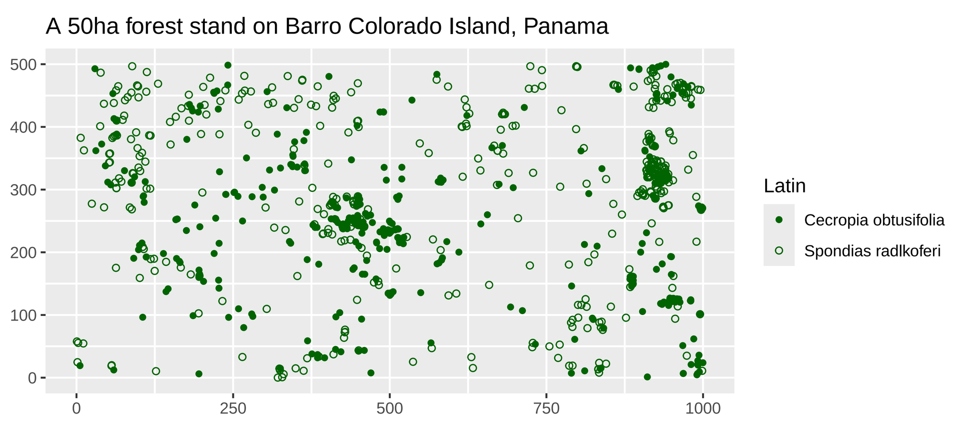
James S. Martin
Imperial College London

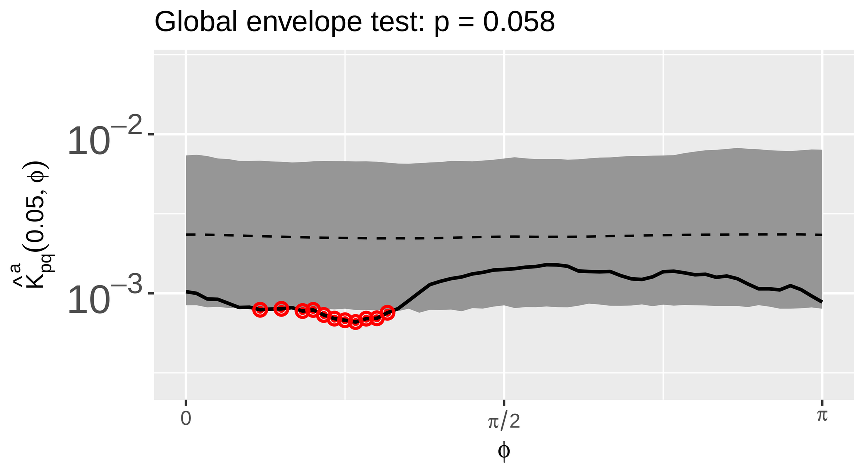
David J. Murrell
University College London

Sofia C. Olhede
École Polytechnique Fédérale de Lausanne

Motivation from Statistical Ecology

Haase (2001) studied clusters of shrubs in dryland ecosystems, and found one species to grow more frequently than expected, to the east of a second species. In preliminary studies of the Barro Colorado Island data, we also found (weak) evidence against isotropy in the interspecific aggregation of *C. obtusifolia* and *S. radlkoferi*, at a range of 50-100m. This is demonstrated below using a global envelope test (significance level 10%), with the null model provided by a bivariate *isotropic* log-Gaussian Cox process; the test statistic is a bivariate version of the sector-K-function, measured over $r \in (0m, 50m)$.





Our model

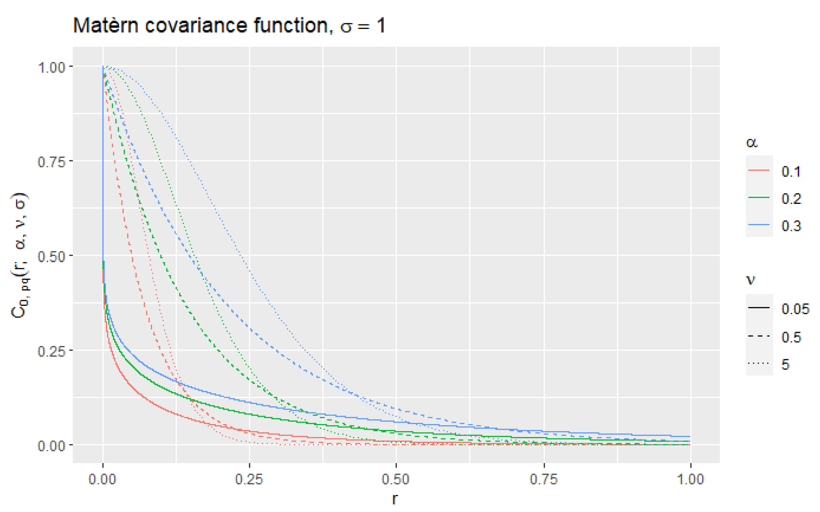
We extend the work of Møller & Toftaker (2014), incorporating geometric anisotropy into the *joint* covariance structure of the driving intensity for a P-variate LGCP. We use a family of deformation matrices $\{\Sigma_{pq}, p, q = 1, \ldots, P\}$ to deform the spatial domains of the auto- and cross-covariance functions, $C_{pq}(h)$, for $h \in \mathbb{R}^2$. We specify the multivariate dependence using a multivariate Matèrn covariance model (Apanasovich et al, 2012):

For $h \in \mathbb{R}^2$, and for $p, q = 1, 2$,

$$C_{pq}(h) = C_{0,pq}(\|\Sigma_{pq}^{-1/2}h\|;\ \alpha_{pq}, \nu_{pq}, \sigma_{pq})$$

$$\Sigma_{pq} = \mathcal{R}_{\theta_{pq}} \begin{pmatrix} 1 & 0 \\ 0 & \zeta_{pq}^2 \end{pmatrix} \mathcal{R}_{\theta_{pq}}^T,$$

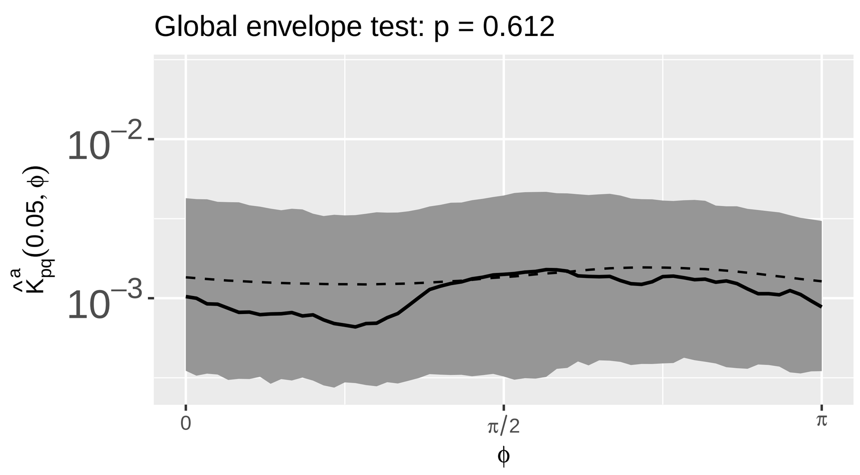
with $\theta_{pq} \in [0, \pi)$ and $\zeta_{pq} \in (0, 1]$ the angle and ratio of anisotropy, respectively.



The technical challenge lies in ensuring validity of the resulting joint-dependence structure: for all $h \in \mathbb{R}^2$, the matrix $(C_{pq}(h))_{p,q=1}^P$ must be nonnegative definite. One key consequence of this requirement is that each zero-lag cross-correlation ρ_{pq} will have an upper bound dependent on $\zeta_{pp}\zeta_{qq}/\zeta_{pq}^2$, with $\zeta_{pq} \in [\zeta_{pp}^{1/2}\zeta_{qq}^{1/2}, 1]$: as the ellipticity of the cross-covariance becomes less pronounced, the maximum possible zero-lag correlation between the two components of the field will decrease.

Application to the BCI data

We fit our model, with $\nu = 0.5$, to the bivariate BCI point pattern above. Using this as our null model, we implement a GET as before, finding insufficient evidence to reject.



In a bivariate Cox process with ***elliptical*** clusters, the ***strength*** of any between-process dependence will depend on the ***geometry*** of each ***marginal*** process.

Take a picture to access the paper

Sufficient Conditions for Model Validity

1. $\exists \Delta_\nu \geq 0$ s.t. $\nu_{pq} - (\nu_{pp} + \nu_{qq})/2 = \Delta_\nu(1 - A_{\nu,pq})$.
2. The matrix with elements $-4\nu_{pq}/\alpha_{pq}^2$ is conditionally nonnegative definite.
3. The matrix with elements
$$\frac{|\Sigma_{pq}|^{1/2}\sigma_{pq}\Gamma(\nu_{pq}+1)}{\pi\,\Gamma\left(\frac{\nu_{pp}+\nu_{qq}}{2}+1\right)\Gamma(\nu_{pq})}\left(\frac{4\nu_{pq}}{\alpha_{pq}^2}\right)^{\Delta_\nu+\frac{\nu_{pp}+\nu_{qq}}{2}}$$
is nonnegative definite.
4. The matrix with elements $-\|\Sigma_{pq}^{1/2}\omega\|^2$ is conditionally nonnegative definite for any $\omega \in \mathbb{R}^2$.

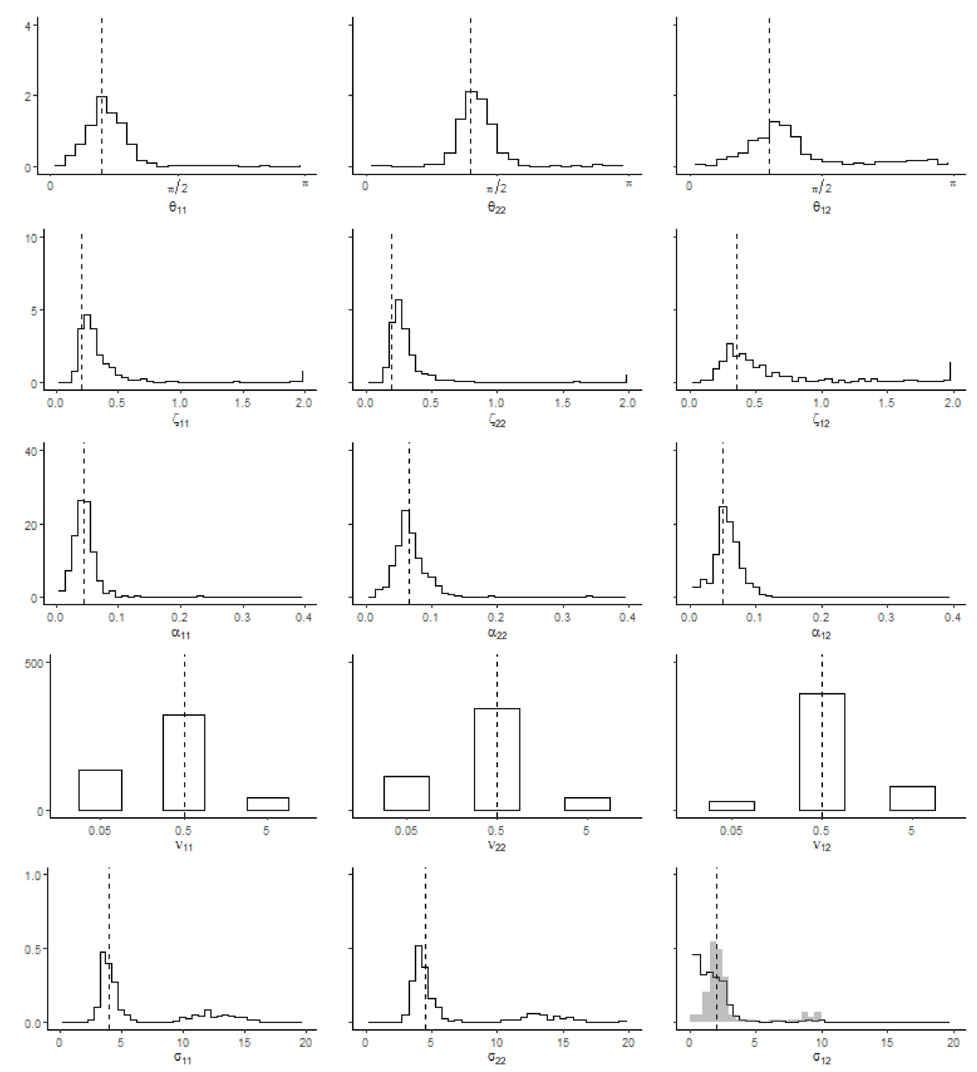
Parameter estimation

We use a two-step procedure:

- Estimate θ_{pp}, ζ_{pp} for $p = 1, \ldots, P$; then estimate θ_{pq}, ζ_{pq} for $p \neq q$.
- Use θ_{pq}, ζ_{pq} to 'isotropize' the data, and estimate the corresponding Mat'ern parameters via MPLE.

We estimate the Matèrn parameters by maximising (a bivariate version of) the Palm-likelihood. This involves a tuning parameter R; we found reasonable performance when setting R approximately equal to the empirical 'range of correlation'.

The following is a proof-of-concept simulation study for our parameter estimation methodology:



References

[1] P. Haase (2001) *J. Veg. Sci.*, **12**, 127-136

[2] T. Apanasovich, M. Genton, & Y. Sun (2012) *JASA*, **107**, 180-193

[3] J. Møller & H. Toftaker (2014) *Scand. J. Stat*, **41**, 414-435